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SHORT TESTS FOR SETS OF LABORATORY WEIGHTS

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ABSTRACT

Three kinds of tests are outlined: First, rough checks for gross errors such as can be detected by simply checking duplicate weights against each other or by comparing a few weights with the sum of those smaller weights whose sum equals the larger weight; second, the comparison with each other of just enough weights and combinations of weights so that the value of each weight can be computed from a standard weight the size of the largest weight in the set; third, the comparison of a sufficient number of weights or combinations so that the agreement of various results will serve as a check against any serious mistake in the observations.

If no standards are available, "relative" values may be found with practically no change in the procedure. The effect of inequality of the arms of the balance beam is eliminated by the method of combining the weighings. Therefore ordinary "direct" methods of weighing may be used.

All multipliers and divisors have been reduced to one figure, and the numbers used in the computations need seldom be larger than three significant figures. Numerical examples illustrate the computations. Every detail of the computations is indicated in full, even though this adds somewhat to the apparent complexity.

An accuracy sufficient for most work can be obtained without detailed corrections for the buoyant effect of the air by attention to the notes given on this subject.

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I. INTRODUCTION

Inaccurate weights often go undiscovered not only through ignorance or lack of facilities, but because of fear that calibration methods are too difficult or too tedious to be practicable in the ordinary laboratory. This article shows two methods for complete calibrations that are believed to be either shorter or much simpler than the calibration schemes generally published.

There is really great need for testing new weights, and much more need for retesting them after some months even when they have not been used. Variations in quality are extreme, and nothing but actual tests of the individual weights themselves will prove either their accuracy or their constancy.

A rough check for gross errors can be made in a very few weighings.

The short calibration, which determines the actual correction for each weight, but without checks on the weighings, requires less than the equivalent of two ordinary direct weighings for each weight, or less than one substitution weighing for each weight.

The second calibration scheme contains very valuable checks for detecting possible mistakes in any of the weighings, and it reduces the effect of ordinary errors of observation. It requires the equivalent of about three ordinary direct weighings for each weight.

The weighings are so combined in both schemes of calibration as to eliminate the effect of inequality of the arms of the balance. Therefore it is possible to use ordinary methods of "direct weighing" in which the object to be weighed or the weight to be tested is placed on the left pan and "weighed" in terms of weights on the right side of the balance.

More accurate methods of weighing may be used if desired, and such use will not in the least change the procedure of the calibrations, but for convenience and brevity weighings will be spoken of as though made by this "direct" method and with the use of a rider.

An important characteristic of these calibrations is the fact that errors or uncertainties in the value of the 100 g standard affect the values of the weights only in proportion to the mass of the weights. Therefore, this source of error has practically no effect on the values of the very small weights. When only relative values are determined by basing them on the 100 g weight of the set, the values for a large part of the set are practically the absolute values because of this reduction in the effect of the error of the 100 g weight on which the values are based.

Since the accuracy of all results depends directly on the reliability of the balance, the second method of calibration may have a great advantage in that it provides some checks on the individual weighings,

giving also some general idea as to the behavior of the balance during the test.

For simplicity of presentation the calibrations are given as applied to a set having the following weights: 100 g, 50 g, 20 g, 10 g, 10 g, 5 g, 2 g, 2 g, 1 g, 500 mg, 200 mg, 100 mg, 100 mg, 50 mg, 20 mg, 10 mg, 10 mg, 5 mg, 2 mg, 2 mg, 1 mg, and 1 mg. From the outline of the calibration and some notes at the end it will be evident that sets having a different arrangement of duplicates can be tested without any difficulty by using the data given for this set.

II. ROUGH CHECKS FOR GROSS ERRORS

Gross errors are so rarely proportional to the mass of the weights that they will be detected almost invariably by checking the different weights of a set against each other.

If only one weight is incorrect it can very quickly be located by a few intercomparisons. First, the 100 g weight is compared with the sum of the weights that make up 100 g, then the 50 g weight with the "summation 50 g," the 5 g weight with the "summation 5 g," and so on down. This will locate the group in which the incorrect weight occurs. The weight can then be located, generally, by checking the two duplicates against each other and the other weight of the group against the proper summation.

If a number of weights are seriously in error, these intercomparisons will merely show that such errors exist, but, in general, it will be necessary to resort to a systematic intercomparison, such as is given below, in order to locate the incorrect weights positively. When such a calibration is made, moreover, it not only locates the weight, but also provides a means of computing the amount of the error.

The method of weighing as well as the care necessary to be exercised must be regulated by the size of the errors involved.

III. THE BALANCE

In the simplest calibrations there is no reduction of observational errors by averaging or similar processes. In fact, errors of different observations are numerically added in many cases. Therefore, both the sensitiveness and the reliability of the balance must be ample for the precision desired. In working downward from larger to smaller weights, as is done here, there is, however, a continual reduction of the errors introduced by the weighings of the larger weights, just as there is a reduction in the effect of any error in the value of the standard. Thus it is quite reasonable to make more accurate observations on the smaller weights, even using a more accurate balance for them. This is often worth while because so many of the small

weights are combined with only one or two large weights in making most weighings.

The reliability of laboratory balances should not be too hastily assumed. The second method of calibration given here contains valuable checks on the observations. While there are too few observations to warrant any close estimate of the accuracy attained, yet these checks will practically always disclose any serious errors. In the shorter calibration, previous experience or special tests must be relied on entirely to determine what accuracy has probably been attained.

Inequality of arms of the beam has no effect because of the way in which observations are combined in these calibration schemes. However, as the approximate effect of such inequality is very easily determined, and may be of use in such work as the rough checks noted above, the following outline is inserted for use in such cases as require it.

First, note the zero reading of the balance. This need not be at the center of the scale. Then place a 100 g weight on each scale pan and find how much must be added to one side of the balance to bring the equilibrium position (rest point) back to the zero reading just noted. Then interchange the 100 g weights and again find how much must be added in order to bring the equilibrium position to the same zero reading. If the added amounts are in both cases on the same side of the balance, their numerical average is the effect sought. If the added amounts are on different sides of the balance, it indicates a relatively large difference between the two weights and the effect of inequality of arms is the algebraic mean or half the numerical difference between the added amounts.

Unless there is some unusual fault in the balance, this effect of unequal arms is proportional to the load. In high-grade "analytical" balances this effect is likely to be anywhere from a few tenths of a milligram to a very few milligrams at a load of 100 g.

If a rider is to be used in the weighings, it is necessary to be sure of the accuracy of its indications. This is true in all methods of calibration, though the fact is often overlooked. A careful inspection is generally enough to determine that the graduations on the beam are equally spaced, for the eye very easily detects such irregularities. The rider should then be placed on the "largest" graduation and checked by a known weight on the scale pan. It should also be checked on the zero graduation, or at the lowest value at which it can be used.

IV. STANDARDS

The complete calibrations, as outlined here, provide for the use of four standards or weights of known value, namely, 100 g, 1 g, 1 mg; and either 5 mg or 10 mg according to the denomination of the rider that

is used. The possibility of dispensing with any or all of these will be seen from the following discussion of their use and from the section on relative values.

The 100 g standard serves as the basis for the values of all of the weights down to 1 g, and the 1 g standard serves similarly for the values of the weights below this. The weights below 1 g could, of course, be based on the newly determined value of the 1 g weight of the set, or the "summation 1,000 mg" could be inserted in the calibration of the second group in place of the 1 g standard. However, the 1 g standard serves as a valuable check on the calibration of the larger weights, and if of platinum it eliminates the need for buoyancy corrections in determining the mass values of platinum fractions of a gram; moreover, in some cases the handling of so many small weights all through the calibration of the larger weights is so objectionable that some other 1 g weight might well be used if a 1 g standard is not available.

In each weighing the weight or weights under test are weighed in terms of some auxiliary weights of the same denomination and the rider. The "auxiliary weights" are not standards and their values need not be known. If weights are not available for this purpose, any ordinary counterpoising material, such as pieces of metal or shot, may be used. It may be found convenient to have the auxiliary weights very slightly lighter than the weights being tested, so that the values determined by the rider will not be negative, but negative values introduce no complications beyond using the sign properly in the calculations.

The 5 mg or 10 mg standard is needed to make a preliminary test of the accuracy of the results given by the rider. If this standard is not available, it is possible first to carry through the calibration as though the rider were exactly correct, thus determining preliminary values for the weights, then to determine a preliminary value for the rider from the preliminary values of the weights tested, and go back over the calibration correcting each weighing for the preliminary correction for the rider if this correction has an appreciable effect on the weighings. If a rider had a very large correction, it might be necessary to repeat this process of recomputation till it produced no further change in the correction for the rider. Such a process is rather complicated, however, which emphasizes the need for this small standard.

The 1 mg standard is used merely as a check on the calibration and accordingly can be dispensed with without such serious harm. In the second scheme of calibration, however, it would be necessary to replace it by some weight in order to secure the proper observations.

The 100 g standard is thus the only one that is indispensable for determining the actual values of the weights of the set. The others are introduced for security and simplicity.

V. NOTATION

Weights will be designated by their nominal value in parentheses, with subscript numbers, to distinguish duplicates. Thus, in a set having two 20 g weights these weights will be designated $(20\text{ g})_1$ and $(20\text{ g})_2$.

To avoid listing the large number of weights used in some groups, the sign Σ will be used for the usual summations used to make up certain amounts, as $\Sigma(5\text{ g})$, $\Sigma(500\text{ mg})$, etc. In all such cases the summation is made up of the fewest weights that can be used, and in the case of duplicates those having the subscript $_1$ are used in preference to those with the subscript $_2$. Thus, $\Sigma(5\text{ g})$ would be $(2\text{ g})_1+(2\text{ g})_2+(1\text{ g})_1$ if there are two 2 g weights and also two 1 g weights.

To avoid a confusing succession of zeros or nines, the values of weights are generally expressed in two terms, the nominal value plus or minus the proper number of milligrams. The second term is then identical with what may best be called "the correction for the weight."

To avoid confusion in thinking it should be carefully noted that this correction has the same algebraic sign as the error of the weight. Thus if the mass of a 20 g weight is too great by 0.5 mg (that is, if it is 0.5 mg too heavy), not only is its mass equal to 20 g + 0.5 mg, or to 20.0005 g, but when it is used in weighing an object, 0.5 mg must be added to the sum of the nominal values of the weights used.

Speaking more strictly, such a designation as $(20\text{ g})_1$ means the mass of this weight, or its actual "value" as distinguished from its nominal value. This meaning of the symbol is seen most clearly in such an equation as

$$(20\text{ g})_1=20\text{ g}+0.5\text{ mg}$$

or

$$(20\text{ g})_1=20.0005\text{ g}$$

The term +0.5 mg is the "correction" for this 20 g weight.

The letter C is used to designate any "auxiliary weight" or counterpoising material such as is discussed above under "standards." The subscript distinguishes the individual weight or combination used.

Each weighing determines the last term in one of the equations given as "observation equations." Thus for the equation

$$(20\text{ g})_1+(20\text{ g})_2+(10\text{ g})=C_{50}+n_2$$

the three weights $(20\text{ g})_1$, $(20\text{ g})_2$, and (10 g) are together weighed in terms of the auxiliary weight C_{50} plus whatever rider reading is

needed to make up the proper amount. Since the auxiliary weight remains the same for two or more weighings and its value cancels out in the computations, the essential feature of each weighing is the determination of the value of n. This value of n will be positive when it is on the right-hand side of the balance and negative when it is on the side with the weights being tested.

VI. GENERAL OUTLINE OF CALIBRATIONS

In both calibrations the set is divided as follows:

	First group	Second group	Third group	Fourth group	Fifth group
(100 g)	(50 g) (20 g) $(10\ g)_1$ $(10\ g)_2$	(5 g) $(2\ g)_1$ $(2\ g)_2$ (1 g)	(500 mg) (200 mg) $(100\ mg)_1$ $(100\ mg)_2$	(50 mg) (20 mg) $(10\ mg)_1$ $(10\ mg)_2$	(5 mg) $(2\ mg)_1$ $(2\ mg)_2$ $(1\ mg)_1$ $(1\ mg)_2$

The first, third, and fourth groups are similar and they are calibrated in exactly the same manner. The second and fifth groups are also similar except that the fifth group has one more weight. The calibrations are arranged, however, so that these two groups are calibrated in the same manner.

Each group is calibrated as a unit, except that the sum of one or more succeeding groups may be used as a fifth weight in each group. Thus in calibrating the first group the sum of the weights of the second group, designated as $\Sigma(10\ g)$, is used as a fifth weight of this first group.

The weighings for each group should, in general, be made as close together as possible in order to allow as little chance as may be for variations in the balance. Some of the exceptions to this rule, of importance in the second calibration method, are noted under that method. The different groups may, however, be separated as completely as desired, which allows parts of the calibration to be done on different days or on different balances.

The computations are shown completely so that it is only necessary to follow the simple equations given. A numerical example of the computations is given at the end of each system of calibration.

VII. SHORTEST SYSTEM OF CALIBRATION

(100 g) and $\Sigma(100\ g)$[1]

Observation equations:

$$(100\ g) = C_{100} + n_1$$
$$(\text{standard } 100\ g) = C_{100} + n_2$$
$$\Sigma(100\ g) = C_{100} + n_3$$

[1] $\Sigma(100\ g)$ must contain the same weights that are included in $\Sigma(50\ g)$, $\Sigma(10\ g)$, and $\Sigma(5\ g)$. See under "notation" above.

Computations: Either equations (1) and (2) or (3) and (4) may be used. The former are perhaps simplest logically, but the latter are the simplest in many ways and are of the form generally used.

$$(100\text{ g}) = (\text{standard }100\text{ g}) + n_1 - n_2 \tag{1}$$

$$\Sigma(100\text{ g}) = (\text{standard }100\text{ g}) + n_3 - n_2 \tag{2}$$

Letting s designate the correction for the standard 100 g weight

$$(100\text{ g}) = 100\text{ g} + s + n_1 - n_2 \tag{3}$$

$$\Sigma(100\text{ g}) = 100\text{ g} + s + n_3 - n_2 \tag{4}$$

$s + n_3 - n_2$ is the correction for $\Sigma(100\text{ g})$ and in the computations below is designated as N.

First group

(50 g) to (10 g) and Σ(10 g)

Observation equations:

$$\begin{aligned}
&(50\text{ g}) && = C_{50} + n_1\\
&(20\text{ g}) + (10\text{ g})_1 + (10\text{ g})_2 + \Sigma(10\text{ g}) && = C_{50} + n_2\\
&(20\text{ g}) + (10\text{ g})_1 && = C_{30} + n_3\\
&(20\text{ g}) \qquad\qquad + (10\text{ g})_2 && = C_{30} + n_4\\
&(20\text{ g}) \qquad\qquad\qquad\qquad + \Sigma(10\text{ g}) && = C_{30} + n_5\\
&\qquad\quad (10\text{ g})_1 + (10\text{ g})_2 + \Sigma(10\text{ g}) && = C_{30} + n_6
\end{aligned}$$

Computations: In preceding computations it was found that

$$\Sigma(100\text{ g}) = (50\text{ g}) + (20\text{ g}) + (10\text{ g})_1 + (10\text{ g})_2 + \Sigma(10\text{ g}) = 100\text{ g} + \text{N}$$

The following results were obtained by a solution combining this equation with the observation equations for this group.

By computing the values of the 20 g and 10 g weights in two steps a large part of the work can be done mentally at a great saving of time and labor. Therefore, this method is given first, even though it may not seem quite so direct and clear.[2] The numerical illustration shows the simplicity of the numerical work and also a convenient way of arranging it.

In this "two step" solution the values of A_1 to A_6 and of 2K are first computed as intermediate quantities. It should also be noted that in this first group the corrections for the last three weights are each computed from the correction for the preceding weight.

[2] For making a first single calibration some persons may find the equations of the "direct" solution enough easier to follow to compensate for the extra work.

$$
\begin{aligned}
A_1 &= n_1 - n_2 \\
A_2 &= n_3 - n_4 \\
A_3 &= n_3 + n_4 \\
A_4 &= n_4 - n_5 \\
A_5 &= n_5 - n_6 \\
A_6 &= n_5 - 3n_6 \\
2\mathrm{K} &= (\mathrm{N} - A_1)
\end{aligned}
$$

$$(50\ \mathrm{g}) = 50\ \mathrm{g} + \frac{1}{2}\ (\mathrm{N} + A_1)$$

$$(20\ \mathrm{g}) = 20\ \mathrm{g} + \frac{1}{5}\ (2\mathrm{K} + A_3 + A_6)$$

Let R_{20} equal the correction for (20 g), which is the value just computed as $\frac{1}{5}\ (2\mathrm{K} + A_3 + A_6)$, then

$$(10\ \mathrm{g})_1 = 10\ \mathrm{g} + \frac{1}{2}\ (R_{20} + A_2 - A_5)$$

Let R_{10} equal the correction just computed for $(10\ \mathrm{g})_1$, then

$$(10\ \mathrm{g})_2 = 10\ \mathrm{g} + (R_{10} - A_2)$$

Let R'_{10} equal the correction just completed for $(10\ \mathrm{g})_2$, then

$$\Sigma(10\ \mathrm{g}) = 10\ \mathrm{g} + (R'_{10} - A_4)$$

The alternative "direct" form of solution is shown in the following equations:

$$(50\ \mathrm{g}) = 50\ \mathrm{g} + \frac{1}{2}\ (\mathrm{N} + n_1 - n_2)$$

$$(20\ \mathrm{g}) = 20\ \mathrm{g} + \frac{1}{5}\ (\mathrm{N} - n_1 + n_2 + n_3 + n_4 + n_5 - 3n_6)$$

$$(10\ \mathrm{g})_1 = 10\ \mathrm{g} + \frac{1}{10}\ (\mathrm{N} - n_1 + n_2 + 6n_3 - 4n_4 - 4n_5 + 2n_6)$$

$$(10\ \mathrm{g})_2 = 10\ \mathrm{g} + \frac{1}{10}\ (\mathrm{N} - n_1 + n_2 - 4n_3 + 6n_4 - 4n_5 + 2n_6)$$

$$\Sigma(10\ \mathrm{g}) = 10\ \mathrm{g} + \frac{1}{10}\ (\mathrm{N} - n_1 + n_2 - 4n_3 - 4n_4 + 6n_5 + 2n_6)$$

The accuracy of the computations can be checked by adding together the final computed corrections for all of the weights of this group, including that for $\Sigma(10\ \mathrm{g})$. The sum should equal N, and is an exact numerical check except for such changes as result from rounding off the results of divisions.

Second group

(5 g) to (1 g)

(S 1 g) is a standard 1 g weight inserted partly as a check, to prevent any mistake in the early part of the work from going too far unnoticed. It also avoids a large amount of handling of the Σ(1,000 mg). The correction for this weight is determined in the calibrations just as though it were an unknown weight.

Observation equations:

$$
\begin{aligned}
&(5\text{ g}) &&= C_5+n_1\\
&(2\text{ g})_1+(2\text{ g})_2+(1\text{ g}) &&= C_5+n_2\\
&(2\text{ g})_1 &&= C_2+n_3\\
&(2\text{ g})_2 &&= C_2+n_4\\
&(1\text{ g})+(\text{S }1\text{ g}) &&= C_2+n_5\\
&(1\text{ g}) &&= C_1+n_6\\
&(\text{S }1\text{ g}) &&= C_1+n_7
\end{aligned}
$$

Computations: The correction for Σ(10 g) was determined in the preceding group, and is designated below by M, giving

$$\Sigma(10\text{ g})=(5\text{ g})+(2\text{ g})_1+(2\text{ g})_2+(1\text{ g})=10\text{ g}+\text{M}$$

This equation has been combined with the observation equations for this group to secure the results given below.

As in the preceding group the computations can be handled more quickly and easily by arranging them in two steps. As in that group the correction for each of the last three weights is computed from the correction for the preceding weight, but the corrections for the 2 g weights are computed last in this group because in this way the computations can be made a little simpler.

$$
\begin{aligned}
A_1&=n_1-n_2 & A_6&=A_2+A_3\\
A_2&=n_3-n_5 & A_7&=2A_4-A_6\\
A_3&=n_4-n_5 & \text{W}&=\frac{1}{2}(\text{M}-A_1)\\
A_4&=n_6-n_7 & &\\
A_5&=n_3-n_4 & &
\end{aligned}
$$

$$(5\text{ g})=5\text{ g}+\frac{1}{2}(\text{M}+A_1)$$

$$(1\text{ g})=1\text{ g}+\frac{1}{5}(\text{W}+A_7)$$

Let R_1 equal the correction for (1 g), then

$$(\text{S }1\text{ g})=1\text{ g}+(R_1-A_4)$$

Let R'_1 equal the correction for (S 1 g), then

$$(2\text{ g})_1=2\text{ g}+(R_1+R'_1+A_2)$$

Let R_2 equal the correction for $(2\text{ g})_1$, then

$$(2\text{ g})_2=2\text{ g}+(R_2-A_5)$$

The alternative "direct" form of solution is shown in the following equations:

$$(5\ \text{g}) = 5\ \text{g} + \frac{1}{2}(\text{M} + n_1 - n_2)$$

$$(2\ \text{g})_1 = 2\ \text{g} + \frac{1}{10}(2\text{M} - 2n_1 + 2n_2 + 6n_3 - 4n_4 - 2n_5 - 2n_6 + 2n_7)$$

$$(2\ \text{g})_2 = 2\ \text{g} + \frac{1}{10}(2\text{M} - 2n_1 + 2n_2 - 4n_3 + 6n_4 - 2n_5 - 2n_6 + 2n_7)$$

$$(1\ \text{g}) = 1\ \text{g} + \frac{1}{10}(\text{M} - n_1 + n_2 - 2n_3 - 2n_4 + 4n_5 + 4n_6 - 4n_7)$$

$$(\text{S}\ 1\ \text{g}) = 1\ \text{g} + \frac{1}{10}(\text{M} - n_1 + n_2 - 2n_3 - 2n_4 + 4n_5 - 6n_6 + 6n_7)$$

If the value calculated for (S 1 g) agrees with its known value,[3] this fact constitutes an excellent check on the correctness of the observations and on the computations down to the point at which (S 1 g) was computed. As in all cases where only a very few observations are taken for each quantity determined, the closeness of the agreement is not a reliable measure of the degree of accuracy attained in the work. The agreement does, however, serve as a check on gross mistakes.

The computations may also be checked by adding together the corrections for all of the weights except (S 1 g). This sum should equal M.

Third, fourth, and fifth groups

(500 mg) to (1 mg)

The weights below 1 g are calibrated by exactly the same processes as the larger ones. The nominal values will be different and the first weighing, corresponding to $(100\ \text{g}) = C_{100} + n_1$ will be omitted with the corresponding first computation equation used to determine (100 g).

The first two weighings will thus be to determine the observation equations:

$$(\text{Standard}\ 1{,}000\ \text{mg}) = C_{1{,}000} + n_2$$

and

$$\Sigma(1{,}000\ \text{mg}) = C_{1{,}000} + n_3$$

Letting s designate the correction for the standard 1 g weight, the correction for $\Sigma(1{,}000\ \text{mg})$, designated by N as in the first group, is given as before by

$$\text{N} = s + n_3 - n_2$$

[3] If (S 1 g) is of platinum or of other material besides brass, proper allowance must be made for the buoyant effect of the air unless the apparent mass as compared with brass standards in air (that is, its "apparent weight in air against brass") is known. This buoyancy correction, to be added to the correction determined in the calibration, would be just the same as in weighing (S 1 g) against a 1 g brass standard.

In the third group the observations and computations are exactly the same as in the first group except that the nominal values 50 g, 20 g, and 10 g will be replaced by 500 mg, 200 mg, and 100 mg. There is, therefore, no need for repeating the equations here. The fourth group will be similarly like the first group, the denominations of the weights tested being in milligrams instead of in grams.

The fifth group will be calibrated like the second group (1 mg)$_2$, being used in place of (S 1 g) in order to have both 1 mg weights in the regular calibration scheme. One of these 1 mg weights may then be checked against the 1 mg standard as a check on the accuracy of the latter part of the calibration.

In testing a set that has only one 1 mg weight, the standard 1 mg would be used in place of (1 mg)$_2$, thus making the calibration still more like that of the second group. If no 1 mg standard is available it will be necessary to put some other 1 mg weight in the calibration temporarily as (1 mg)$_2$ because it is impossible to obtain a solution of the equations without n_5 and n_7.

Example of Computations

SHORTEST SYSTEM OF CALIBRATION

(100 g) and Σ(100 g)

$s = -0.3$ mg

Observed	Computed
$n_1 = +0.5$ mg	$s = -0.3$
	$n_1 - n_2 = +0.4$
$n_2 = +0.1$ mg	Sum $= +0.1$ mg = Correction for (100 g)
$n_3 = +0.0$ mg	$s = -0.3$
	$n_3 - n_2 = -0.1$
	Sum $= -0.4$ mg = Correction for $\Sigma(100\text{ g}) = N$

***First group*: (50 g) to (10 g) and Σ(10 g)**

Observed	Computed (intermediate step)
$n_1 = +0.4$ mg	$n_1 - n_2 = +0.3$ mg $= A_1$
$n_2 = +0.1$ mg	$n_3 - n_4 = +0.1$ mg $= A_2$
$n_3 = +0.8$ mg	$n_3 + n_4 = +1.5$ mg $= A_3$
$n_4 = +0.7$ mg	$n_4 - n_5 = +0.6$ mg $= A_4$
$n_5 = +0.1$ mg	$n_5 - n_6 = -0.1$ mg $= A_5$
$n_6 = +0.2$ mg	$n_5 - 3n_6 = -0.5$ mg $= A_6$

Computed corrections

$$\mathrm{N}+A_1=-0.1$$
$$\tfrac{1}{2}(\mathrm{N}+A_1)=\quad 0.0 \text{ mg}=\text{Correction for (50 g)}$$
$$\mathrm{N}-A_1=-0.7 \qquad =2\mathrm{K}$$
$$\underline{A_3+A_6=+1.0}$$
$$\underline{\text{Sum}=+0.3}$$
$$\tfrac{1}{5}\text{ sum}=+0.0_6 \text{ mg}=\text{Correction for (20 g)}$$
$$\underline{A_2-A_5=+0.2}$$
$$\underline{\text{Sum}=+0.2_6}$$
$$\tfrac{1}{2}\text{ sum}=+0.1 \text{ mg}=\text{Correction for }(10\text{ g})_1=R_{10}$$
$$\underline{A_2=+0.1}$$
$$R_{10}-A_2=\quad 0.0 \text{ mg}=\text{Correction for }(10\text{ g})_2=R'_{10}$$
$$\underline{A_4=+0.6}$$
$$R'_{10}-A_4=-0.6 \text{ mg}=\text{Correction for }\Sigma(10\text{ g})=\mathrm{M}$$

Check on computations: Sum of corrections for (50 g) to Σ(10 g), inclusive,

$$0.0+0.1+0.1+0.0-0.6=-0.4=\mathrm{N}$$

Second group: (5 g) to (1 g)

Observed	Computed (intermediate step)
$n_1=+0.4$ mg	$n_1-n_2=-0.4 \text{ mg}=A_1$
$n_2=+0.8$ mg	$n_3-n_5=-0.6 \text{ mg}=A_2$
$n_3=\quad 0.0$ mg	$n_4-n_5=-0.2 \text{ mg}=A_3$
$n_4=+0.4$ mg	$n_6-n_7=+0.1 \text{ mg}=A_4$
$n_5=+0.6$ mg	$n_3-n_4=-0.4 \text{ mg}=A_5$
$n_6=+0.6$ mg	$A_2+A_3=-0.8 \text{ mg}=A_6$
$n_7=+0.5$ mg	$2A_4-A_6=+1.0 \text{ mg}=A_7$

Computed corrections

$$\mathrm{M}+A_1=-1.0$$
$$\tfrac{1}{2}(\mathrm{M}+A_1)=-0.5 \text{ mg}=\text{Correction for (5 g)}$$
$$\underline{\mathrm{M}-A_1=-0.2}$$
$$\tfrac{1}{2}(\mathrm{M}-A_1)=-0.1 \qquad =\mathrm{W}$$
$$\underline{A_7=+1.0}$$
$$\underline{\mathrm{W}+A_7=+0.9}$$
$$\tfrac{1}{5}(\mathrm{W}+A_7)=+0.2 \text{ mg}=\text{Correction for (1 g)}=R_1$$
$$\underline{A_4=+0.1}$$
$$R_1-A_4=+0.1 \text{ mg}=\text{Correction for (S 1 g)}=R'_1$$
$$R'_1+R_1=+0.3$$
$$\underline{A_2=-0.6}$$
$$\text{Sum}=-0.3 \text{ mg}=\text{Correction for }(2\text{ g})_1=R_2$$
$$\underline{A_5=-0.4}$$
$$R_2-A_5=+0.1 \text{ mg}=\text{Correction for }(2\text{ g})_2$$

Check on computations: Sum of corrections for all except (S 1 g):

$$-0.5+0.2-0.3+0.1=-0.5;\ M=-0.6$$

The apparent discrepancy is the effect of rounding off the value for (1 g).

Previously known correction for (S 1 g) = +0.05 mg. This checks the value found to within the accuracy to which this work is done.

VIII. SECOND SYSTEM OF CALIBRATION

Even careful skilled observers make many more mistakes than most people realize. Therefore, it is important to have reliable checks in determining fundamental data, such as the values of weights. In order to secure such checks it is necessary to make more observations than just enough to compute the values of the weights, as was done in the preceding scheme.

The additional weighings are of double value, however, because the increase in the number of observations raises the final accuracy somewhat, in addition to checking against mistakes.

Most of the checks provided are better than mere repetition of weighings because they involve entirely different combinations so that there is little chance for repeating the same mistake.

For the weighings of (100 g), Σ(100 g), Σ(1,000 mg), and for the first two weighings in each group, (50 g) and Σ(50 g), (5 g), and Σ(5 g), etc., there are no satisfactory checks except to repeat the weighings. For many reasons it is better to repeat these weighings after all of the other weighings have been made. Special effort to avoid repeating any possible mistake should be made both by extra care and by varying the weighing procedure if this is possible. These repeated weighings, although listed as parts of the various "groups," are computed separately, and there is no harm in making them on a different day or even on a different balance from that used for the first observations.

(100 g) and Σ(100 g)[4]

Observation equations: These are just the same as in the shortest calibration.

$$(100\text{ g})=C_{100}+n_1$$

$$(\text{Standard }100\text{ g})=C_{100}+n_2$$

$$\Sigma(100\text{ g})=C_{100}+n_3$$

These first three weighings should be made at the beginning of the calibration and repeated, as already explained, after the calibration has been carried through to the end of the set.

[4] Σ(100 g) must contain the same weights that are included in Σ(50 g), Σ(10 g), and Σ(5 g). See under "notation" above.

Computations: It is, perhaps, best to calculate the values of (100 g) and $\Sigma(100\text{ g})$ from these first weighings by themselves and then from the repeated weighings separately. The average values would then be used in the further calculations.

Letting s designate the correction for the standard 100 g weight, we have, as before,

$$(100\text{ g}) = 100\text{ g} + s + n_1 - n_2$$
$$\Sigma(100\text{ g}) = 100\text{ g} + s + n_3 - n_2$$

$s + n_3 - n_2$ is the correction for $\Sigma(100\text{ g})$, and is designated by N in the computations below.

First group

(50 g) to (10 g) and Σ(10 g)

Observation equations: The following observation equations require the same weighings as were used in the short calibration with the addition of those for the last four equations. The first two observations should be repeated at the close of the calibration of the set, as already indicated.

$$\begin{array}{lllll}
(50\text{ g}) & & & & = C_{50} + n_1 \\
(20\text{ g}) + & (10\text{ g})_1 + & (10\text{ g})_2 + & \Sigma(10\text{ g}) & = C_{50} + n_2 \\
(20\text{ g}) + & (10\text{ g})_1 & & & = C_{30} + n_3 \\
(20\text{ g}) & & + (10\text{ g})_2 & & = C_{30} + n_4 \\
(20\text{ g}) & & & + \Sigma(10\text{ g}) & = C_{30} + n_5 \\
 & (10\text{ g})_1 + & (10\text{ g})_2 + & \Sigma(10\text{ g}) & = C_{30} + n_6 \\
 & & (10\text{ g})_2 + & \Sigma(10\text{ g})^* & = C_{20}^* + n_7 \\
 & (10\text{ g})_1 & & + \Sigma(10\text{ g}) & = C_{20} + n_8 \\
 & (10\text{ g})_1 + & (10\text{ g})_2 & & = C_{20} + n_9 \\
(20\text{ g}) & & & & = C_{20} + n_{10}
\end{array}$$

Computations and checks: The values of (50 g) and of K may be computed from the first two weighings and then from the repeated weighings, separately. The average values would then be used in further work. This computation may be checked by the fact that the sum of K and the correction for (50 g) equals N.

The following check on the other observations may be noted before computing the values of the other weights.

$$n_3 + n_7 = n_4 + n_8 = n_5 + n_9 = n_6 + n_{10}$$

* If the deflection of the pointer is being used in the weighings, instead of bringing the beam to the zero point by small weights or the rider, there is an advantage in working at as constant a load as possible in order to keep the sensibility constant. To do this an extra 10 g counterpoise can be placed on both sides of the balance during the last four weighings. When this is done the value of the counterpoise on the left pan can be computed, if desired, as follows:

$$C_L = 10\text{ g} + \frac{1}{2}\text{ [correction for (2 g)]} - \frac{1}{4}(A + B + C + D)$$

These sums should be equal within twice the maximum error to be expected in each weighing.

The values of the 20 g and 10 g weights are computed in two steps somewhat similar to those used in the shorter calibration.

$$(50\text{ g}) = 50\text{ g} + \frac{1}{2}(\text{N} + n_1 - n_2)$$

$$\text{K} = \frac{1}{2}(\text{N} - n_1 + n_2)$$

$$A = n_3 - n_7$$

$$B = n_4 - n_8$$

$$C = n_5 - n_9$$

$$D = n_6 - n_{10}$$

$$(20\text{ g}) = 20\text{ g} + \frac{1}{10}(4\text{ K} + A + B + C - 3\,D)$$

$$(10\text{ g})_1 = 10\text{ g} + \frac{1}{10}(2\text{ K} + 3\,A - 2\,B - 2\,C + D)$$

Let R equal the correction for $(10\text{ g})_1$ just computed, then

$$(10\text{ g})_2 = 10\text{ g} + R - \frac{1}{2}(A - B)$$

$$\Sigma(10\text{ g}) = 10\text{ g} + R - \frac{1}{2}(A - C)$$

The alternative "direct" form of solution is given in the following equations, which have about the same advantages and disadvantages indicated in connection with those for the first group of the shorter system of calibration. The two forms of solution give identical results, and are the result of a least square solution [6] of the observation equations combined with the equation

$$(50\text{ g}) + (20\text{ g}) + (10\text{ g})_1 + (10\text{ g})_2 + \Sigma(10\text{ g}) = 100\text{ g} + \text{N}$$

which was determined in the preceding group.

$$(50\text{ g}) = 50\text{ g} + \tfrac{1}{2}(\text{N} + n_1 - n_2)$$

$$(20\text{ g}) = 20\text{ g} + \frac{1}{10}(2\text{N} - 2n_1 + 2n_2 + n_3 + n_4 + n_5 - 3n_6 - n_7 - n_8 - n_9 + 3n_{10})$$

$$(10\text{ g})_1 = 10\text{ g} + \frac{1}{10}(\text{N} - n_1 + n_2 + 3n_3 - 2n_4 - 2n_5 + n_6 - 3n_7 + 2n_8 + 2n_9 - n_{10})$$

$$(10\text{ g})_2 = 10\text{ g} + \frac{1}{10}(\text{N} - n_1 + n_2 - 2n_3 + 3n_4 - 2n_5 + n_6 + 2n_7 - 3n_8 + 2n_9 - n_{10})$$

$$\Sigma(10\text{ g}) = 10\text{ g} + \frac{1}{10}(\text{N} - n_1 + n_2 - 2n_3 - 2n_4 + 3n_5 + n_6 + 2n_7 + 2n_8 - 3n_9 - n_{10})$$

---

[6] The complete least square solution gives values for C_{50}, C_{30}, and C_{20}. These are not true values, however, but what may be called effective values; that is, they are the values of weights, placed on the same scale pan as the weights being tested, and of the proper value to just counterbalance C_{50}, C_{30}, and C_{20}. Since the ratio of the arms of the balance beam is likely to change slightly with temperature or with age and use, these values would be subject to the same changes.

The primary check on the computation of the values for the 20 g and 10 g weights is the fact that the sum of the computed corrections, including the correction for $\Sigma(10\text{ g})$, must equal the value of K. This check is numerically exact and the only discrepancy should be that arising from dropping doubtful figures during the computation.

A second possible check on the computations, and one that may be of value in locating a mistake, is contained in the following equations in which $Cr(20\text{ g})$, $Cr(10\text{ g})_1$, etc., refer to the computed corrections for the indicated weights. This check is similarly exact.

$$Cr(10\text{ g})_1 - Cr(10\text{ g})_2 = \frac{1}{2}(A-B)$$

$$Cr(10\text{ g})_2 - Cr\Sigma(10\text{ g}) = \frac{1}{2}(B-C)$$

$$Cr(20\text{ g}) - Cr(10\text{ g})_1 - Cr(10\text{ g})_2 = \frac{1}{2}(C-D)$$

Second group

(5 g) to (1 g)

In this group all but the first two weighings are entirely different from those used in the short calibration. The 1 g weight of the set may be used as $(1\text{ g})_1$. The second 1 g weight $(1\text{ g})_2$ is any other weight or counterpoising material and is inserted for several reasons, such as to obtain a calibration that has better checks and is easier to compute, to allow the second and fifth groups to be calibrated in exactly the same manner, to make more weighings at a constant load, etc. The $\Sigma(1{,}000\text{ mg})$ may be used for $(1\text{ g})_2$ if the extra handling of these small weights is not considered too objectionable. If this summation is determined here, the weighings of (S 1 g) and $\Sigma(1{,}000\text{ mg})$ just before the third group can be omitted, with their repetitions at the close of the calibration.

(S 1 g) is the 1 g standard. Its value is determined just as though it were an unknown weight, and the difference between its calculated and its known correction gives a valuable check on the accuracy of the calibration down to this point.

The last weighing is the same as the third. In this case the repeated weighing should be made immediately after the others, however, as it is needed in the same group of computations in order to have two independent determinations of this quantity. It also serves as a valuable check on the constancy of the balance during these last eight weighings.

As with the first two weighings of the preceding group, the weighing of (5 g) and $\Sigma(5\text{ g})$ should be repeated at the close of the calibration of the set.

Observation equations:

$$
\begin{array}{llllll}
(5\text{ g}) & & & & & = C_5 + n_1 \\
(2\text{ g})_1 + (2\text{ g})_2 + (1\text{ g})_1 & & & & & = C_5 + n_2 \\
& & & (1\text{ g})_1 + (1\text{ g})_2 + (\text{S }1\text{ g}) & & = C_3 + n_3 \\
(2\text{ g})_1 & & & + (\text{S }1\text{ g}) & & = C_3 + n_4 \\
& (2\text{ g})_2 & & + (\text{S }1\text{ g}) & & = C_3 + n_5 \\
& (2\text{ g})_2 & + (1\text{ g})_2 & & & = C_3 + n_6 \\
(2\text{ g})_1 & & + (1\text{ g})_2 & & & = C_3 + n_7 \\
(2\text{ g})_1 & + (1\text{ g})_1 & & & & = C_3 + n_8 \\
& (2\text{ g})_2 + (1\text{ g})_1 & & & & = C_3 + n_9 \\
& & & (1\text{ g})_1 + (1\text{ g})_2 + (\text{S }1\text{ g}) & & = C_3 + n_{10}
\end{array}
$$

Computations and checks: As with (50 g) and K of the preceding group, the values of (5 g) and W may be computed from the first two weighings and then from the repeated weighings separately and the average values taken. This computation may be checked by the fact that

$$Cr(5\text{ g}) + \text{W} = Cr\Sigma(10\text{ g})$$

n_3 should equal n_{10} within the error of observation, and this is the only direct check on these two weighings.

The checks on the other weighings are found in the "intermediate" quantities X_2, X_3, and Z, below. The various values of each should be equal within about three or four times the error of observation. The average value of each is then used in the later computation.

The correction for Σ(10 g), found in the preceding group is designated by the letter M.

$$(5\text{ g}) = 5\text{ g} + \frac{1}{2}(\text{M} + n_1 - n_2)$$

$$\text{W} = \frac{1}{2}(\text{M} - n_1 + n_2)$$

$$X_1 = (n_3 + n_8 + n_9 + n_{10}) - (n_4 + n_5 + n_6 + n_7)$$

$$X_2 = n_7 - n_8 = n_6 - n_9$$

$$X_3 = n_4 - n_8 = n_5 - n_9$$

$$Z = n_4 - n_5 = n_7 - n_6 = n_8 - n_9$$

$$(2\text{ g})_1 = 2\text{ g} + \frac{1}{10}(4\text{ W} - X_1 + 5\ Z)$$

$$(2\text{ g})_2 = 2\text{ g} + \frac{1}{10}(4\text{ W} - X_1 - 5\ Z)$$

$$(1\text{ g})_1 = 1\text{ g} + \frac{1}{5}(\text{W} + X_1)$$

Let R_1 equal the correction for $(1\text{ g})_1$ just computed, then

$$(1\text{ g})_2 = 1\text{ g} + (R_1 + X_2)$$

$$(\text{S }1\text{ g}) = 1\text{ g} + (R_1 + X_3)$$

The alternative "direct" form of solution [7] is as follows. This is also a least square solution of the observation equations combined with the equation

$$(5\text{ g})+(2\text{ g})_1+(2\text{ g})_2+(1\text{ g})=10\text{ g}+\text{M}$$

which was determined in the preceding group.

$$(5\text{ g})=5\text{ g}+\frac{1}{2}(\text{M}+n_1-n_2)$$

$$(2\text{ g})_1=2\text{ g}+\frac{1}{10}(2\text{M}-2n_1+2n_2-n_3+\frac{8}{3}n_4-\frac{2}{3}n_5-\frac{2}{3}n_6+\frac{8}{3}n_7+\frac{2}{3}n_8-\frac{8}{3}n_9-n_{10})$$

$$(2\text{ g})_2=2\text{ g}+\frac{1}{10}(2\text{M}-2n_1+2n_2-n_3-\frac{2}{3}n_4+\frac{8}{3}n_5+\frac{8}{3}n_6-\frac{2}{3}n_7-\frac{8}{3}n_8+\frac{2}{3}n_9-n_{10})$$

$$(1\text{ g})_1=1\text{ g}+\frac{1}{10}(\text{M}-n_1+n_2+2n_3-2n_4-2n_5-2n_6-2n_7+2n_8+2n_9+2n_{10})$$

$$(1\text{ g})_2=1\text{ g}+\frac{1}{10}(\text{M}-n_1+n_2+2n_3-2n_4-2n_5+3n_6+3n_7-3n_8-3n_9+2n_{10})$$

$$(\text{S 1 g})=1\text{ g}+\frac{1}{10}(\text{M}-n_1+n_2+2n_3+3n_4+3n_5-2n_6-2n_7-3n_8-3n_9+2n_{10})$$

The following checks on the computations should be exact except for the effect of rounding off when dropping the doubtful figures.

Using $Cr(2\text{ g})_1$, $Cr(2\text{ g})_2$, etc., to indicate the corrections computed for the weights,

$Cr(2\text{ g})_1+Cr(2\text{ g})_2+Cr(1\text{ g})_1=\text{W}$
$Cr(2\text{ g})_1-Cr(2\text{ g})_2=Z$ (note algebraic signs especially).
$Cr(1\text{ g})_2-Cr(1\text{ g})_1=X_2$
$Cr(\text{S 1 g})-Cr(1\text{ g})_1=X_3$

Third, fourth, and fifth groups

(500 mg) to (1 mg)

The weights below 1 g are calibrated by exactly the same process as the larger ones. The nominal values will be different, the first weighing corresponding to $(100\text{ g})=C_{100}+n_1$ will be omitted, and the first computation equation will not be needed.

The first two weighings will thus determine the equations

$$(\text{Standard }1{,}000\text{ mg})=C_{1,000}+n_2$$

and

$$\Sigma(1{,}000\text{ mg})=C_{1,000}+n_3$$

[7] For note on values for C_5 and C_3 see footnote to solution for the first group.

Letting s designate the correction for the standard 1 g weight, the correction for Σ(1,000 mg), designated by N as in the first group, is given by

$$N = s + n_3 - n_2$$

In the third group the observations and computations are exactly the same as in the first group except that the nominal values 50 g, 20 g, and 10 g will be replaced in every case by 500 mg, 200 mg, and 100 mg. There is therefore no need for repeating the equations here. The fourth group will, of course, be like the first instead of the second group, and thus the fourth and fifth groups will be exactly like the first and second groups except that the denomination will be milligrams instead of grams.

If a set has only one 1 mg weight, it will be necessary to put in some other weight for $(1\text{ mg})_2$, since the omission of this weight would require an entirely different form of calibration.

Example of Computations

SECOND SYSTEM OF CALIBRATION

(100 g) and Σ(100 g)

$s = +0.3$ mg

Observed

	First	Last
$n_1 =$	$+0.1$ mg	$+0.1$ mg
$n_2 =$	$+0.3$ mg	$+0.3$ mg
$n_3 =$	$+3.0$ mg	$+3.2$ mg

Computed

From first observations:

$$s = +0.3$$
$$n_1 - n_2 = -0.2$$
$$\text{Sum} = +0.1$$

From last observations:

$$s = +0.3$$
$$n_1 - n_2 = -0.2$$
$$\text{Sum} = +0.1$$

Average $= +0.1$ mg = Correction for (100 g)

From first observations:

$$s = +0.3$$
$$n_3 - n_2 = +2.7$$
$$\text{Sum} = +3.0$$

From last observations:

$$s = +0.3$$
$$n_3 - n_2 = +2.9$$
$$\text{Sum} = +3.2$$

Average $= +3.1$ mg = Correction for $\Sigma(100\text{ g}) = N$

First group: (50 g) to (10 g) and Σ(10 g)

Observed

	Repeated
$n_1 = +0.6$ mg	$+0.4$
$n_2 = +2.4$ mg	$+2.1$
$n_3 = +1.4$ mg	
$n_4 = +1.0$ mg	
$n_5 = +1.3$ mg	
$n_6 = +1.6$ mg	
$n_7 = +0.9$ mg	
$n_8 = +1.2$ mg	
$n_9 = +1.0$ mg	
$n_{10} = +0.9$ mg	

Checks on observations:

$n_3 + n_7 = +2.3$

$n_4 + n_8 = +2.2$

$n_5 + n_9 = +2.3$

$n_6 + n_{10} = +2.5$

This agreement is satisfactory if the weighings can be made to only about 0.1 mg.

Computed

From first observations:

$$N = +3.1$$
$$n_1 - n_2 = -1.8$$
$$\text{Sum} = +1.3$$
$$\tfrac{1}{2} \text{ sum} = +0.6_5$$

From repeated observations:

$$N = +3.1$$
$$n_1 - n_2 = -1.7$$
$$\text{Sum} = +1.4$$
$$\tfrac{1}{2} \text{ sum} = +0.7_0$$

Average $= +0.7$ mg = Correction for (50 g)

From first observations:

$$N - (n_1 - n_2) = +4.9$$
$$\tfrac{1}{2} \text{ difference} = +2.4_5$$

From repeated observations:

$$N - (n_1 - n_2) = +4.8$$
$$\tfrac{1}{2} \text{ difference} = +2.4_0$$

Average $= +2.4$ mg = Correction for Σ(50 g) = K

Check on computation: The sum of the corrections for (50 g) and Σ(50 g)

$$+0.7 + 2.4 = +3.1 = N.$$

$$n_3 - n_7 = +0.5 = A$$
$$n_4 - n_8 = -0.2 = B$$
$$n_5 - n_9 = +0.3 = C$$
$$n_6 - n_{10} = +0.7 = D$$

$+4K =$	$+9.6$	
$+A =$	$+0.5$	
$+B =$		-0.2
$+C =$	$+0.3$	
$-3D =$		-2.1
	$+10.4$	-2.3
Sum =	$+8.1$	
$\frac{1}{10}$ sum =	$+0.8$ mg = Correction for (20 g)	

$+2K =$	$+4.8$	
$+3A =$	$+1.5$	
$-2B =$	$+0.4$	
$-2C =$		-0.6
$+D =$	$+0.7$	
	$+7.4$	-0.6
Sum =	$+6.8$	
$\frac{1}{10}$ sum =	$+0.7$ mg = Correction for $(10 \text{ g})_1 = R$	

$$
\begin{aligned}
R &= +0.7\\
\tfrac{1}{2}(A-B) &= +0.4\\
\hline
R-\tfrac{1}{2}(A-B) &= +0.3\text{ mg} = \text{Correction for } (10\text{ g})_2
\end{aligned}
$$

$$
\begin{aligned}
R &= +0.7\\
\tfrac{1}{2}(A-C) &= +0.1\\
\hline
R-\tfrac{1}{2}(A-C) &= +0.6\text{ mg} = \text{Correction for } \Sigma(10\text{ g}) = \mathrm{M}
\end{aligned}
$$

Checks on computations: Sum of corrections for weights (20 g) and (10 g).

$$+0.8+0.7+0.3+0.6 = +2.4 = \mathrm{K}$$

Also

$$
\begin{aligned}
Cr(10\text{ g})_1 - Cr(10\text{ g})_2 &= +0.4 = {}^*1/2(A-B) = +0.3_5\\
Cr(10\text{ g})_2 - Cr\Sigma(10\text{ g}) &= -0.3 = {}^*1/2(B-C) = -0.2_5\\
Cr(20\text{ g}) &= +0.8\\
-Cr(10\text{ g})_1 &= -0.7\\
-Cr(10\text{ g})_2 &= -0.3\\
\hline
Cr(20\text{ g}) - Cr(10\text{ g})_1 - Cr(10\text{ g})_2 &= -0.2 = 1/2(C-D) = -0.2
\end{aligned}
$$

Second group: (5 g) to (1 g)

Observed	Repeated	Computed
$n_1 = +0.2$	0.0	From first observations:
$n_2 = +0.7$	$+0.8$	$\mathrm{M} = +0.6$
$n_3 = 0.0$		$(n_1 - n_2) = -0.5$
$n_4 = +0.6$		Sum $= +0.1$
$n_5 = +0.4$		1/2 sum $= +0.05$
$n_6 = +0.1$		From repeated observations:
$n_7 = +0.1$		$\mathrm{M} = +0.6$
$n_8 = +0.6$		$(n_1 - n_2) = -0.8$
$n_9 = +0.6$		Sum $= -0.2$
$n_{10} = 0.0$		1/2 sum $= -0.10$
For checks on observations note: $n_3 = n_{10}$ and the "differences" below, under X_2, X_3, and Z are within the observational error.		Average $= -0.02$ mg = Correction for (5 g)
		From first observations:
		$\mathrm{M} - (n_1 - n_2) = +1.1$
		1/2 difference $= +0.55$
		From repeated observations:
		$\mathrm{M} - (n_1 - n_2) = +1.4$
		1/2 difference $= +0.70$
		Average $= +0.62$ mg = Correction for $\Sigma(5\text{ g}) = \mathrm{W}$

Check: $Cr(5\text{ g}) + Cr\Sigma(5\text{ g}) = -0.02 + 0.62 = +0.6 = \mathrm{M}$

* In both cases the agreement is within the effect of dropping doubtful figures.

$$
\begin{array}{rrr}
+n_3 = & 0.0 & \\
-n_4 = & & -0.6 \\
-n_5 = & & -0.4 \\
-n_6 = & & -0.1 \\
-n_7 = & & -0.1 \\
+n_8 = & +0.6 & \\
+n_9 = & +0.6 & \\
+n_{10} = & 0.0 & \\
\hline
 & +1.2 & -1.2 \\
\text{Sum} = & 0.0 & = X_1
\end{array}
$$

Difference = 0.0 mg

$$
\begin{array}{rl}
n_7 - n_8 = & -0.5 \\
n_6 - n_9 = & -0.5 \\
\hline
\text{Average} = & -0.5 = X_2
\end{array}
$$

Difference = 0.2 mg

$$
\begin{array}{rl}
n_4 - n_8 = & 0.0 \\
n_5 - n_9 = & -0.2 \\
\hline
\text{Average} = & -0.1 = X_3
\end{array}
$$

Maximum difference = 0.2 mg

$$
\begin{array}{rl}
n_4 - n_5 = & +0.2 \\
n_7 - n_6 = & 0.0 \\
n_8 - n_9 = & 0.0 \\
\hline
\text{Average} = & +0.07 = Z
\end{array}
$$

$$
\begin{array}{rl}
4\,\mathrm{W} - X_1 = & +2.48 \\
5\,Z = & +0.35 \\
\hline
1/10\ \text{sum} = & +0.28\ \text{mg} = \text{Correction for } (2\ \mathrm{g})_1 \\
1/10(4\,\mathrm{W} - X_1 - 5\,Z) = & +0.21\ \text{mg} = \text{Correction for } (2\ \mathrm{g})_2 \\
\mathrm{W} + X_1 = & +0.62 \\
1/5(\mathrm{W} + X_1) = & +0.12\ \text{mg} = \text{Correction for } (1\ \mathrm{g})_1 = R_1 \\
R_1 + X_2 = & -0.38\ \text{mg} = \text{Correction for } (1\ \mathrm{g})_2 \\
R_1 + X_3 = & +0.02\ \text{mg} = \text{Correction for } (\mathrm{S}\ 1\ \mathrm{g})
\end{array}
$$

Checks on computations:

$$Cr(2\ \mathrm{g})_1 + Cr(2\ \mathrm{g})_2 + (1\ \mathrm{g})_1 = +0.28 + 0.21 + 0.12 = +0.61;\ \mathrm{W} = +0.62$$

The agreement is within the effect of rounding off values during the computations.

$$
\begin{array}{rl}
Cr(2\ \mathrm{g})_1 - Cr(2\ \mathrm{g})_2 = & +0.28 - 0.21 = +0.07 = Z \\
Cr(1\ \mathrm{g})_2 - Cr(1\ \mathrm{g})_1 = & -0.38 - 0.12 = -0.50 = X_2 \\
Cr(\mathrm{S}\ 1\ \mathrm{g}) - Cr(1\ \mathrm{g})_1 = & +0.02 - 0.12 = -0.10 = X_3
\end{array}
$$

Previously known correction for (S 1 g) = +0.04 mg. This is equal to the value found in the calibration, namely, +0.02 mg, within the error of observation.

IX. VARIATIONS FOR DIFFERENT KINDS OF SETS

A number of sets of weights have only one weight of 2 g and three weights of 1 g, with the same arrangement in the milligrams. To calibrate such sets these groups are calibrated just like the first group, with proper changes in the nominal values, just as these nominal values were changed in calibrating the weights below 1 g. The third 1 g or 1 mg weight would be used in place of Σ(10 g). Then to secure a check against the standards it would be necessary to make extra weighings. Similarly, sets having two 20 g weights would be calibrated by using for the group from 50 g to 10 g, the scheme outlined for 5 g to 1 g, except that Σ(10 g) would take the place of (S 1 g), and that in the second form of calibration an extra 10 g weight must be inserted as $(10\ g)_2$.

In general, the two groups of observation and computation equations may be used for any denominations whatever, provided the denominations have the same relative values to each other as do the weights for which the equations are given, or, in other words, the nominal values in any group may be multiplied by any number whatever, provided the same multiplier is used throughout the group.

X. DETERMINATION OF "RELATIVE VALUES"

While it is true that many kinds of work need only relative values, yet laboratory weights are so universally based on the true gram, and complications or errors are so likely to occur when they do not have their true values, that it is seldom advisable to determine anything but actual values if this can be done.

If standards are not available, relative values may be obtained by exactly the same process of calibration, by simply using the 100 g weight of the set as the standard, assuming its correction to be zero. In starting this calibration the weighing of (S 100 g) would be omitted, leaving the determination of Σ(100 g) exactly like the determination of Σ(1,000 mg).

The values of all the smaller weights, as obtained in such a calibration, are practically the actual values because the effect of the error of the 100 g weight is reduced in proportion to the mass of the weight tested. Relative values based on the 100 g weight may be reduced easily to actual values if the value of the 100 g weight is determined later, by adding to the "relative corrections" the proper proportional part of the correction for this weight.

In determining relative values by any method of calibration, a fact likely to be overlooked is that the differences between the various weights must be determined in terms of the same weight that is used as the basis of the calibration. While making the necessary weighings in such a "relative" calibration it is impossible to know whether the rider values used are the correct values in terms of this basic

weight; therefore, any calibration for relative values involves the situation noted under the section on standards when 5 mg and 10 mg standards are not available. In such cases the rider may be assumed to give correct results till the calibration has been completed and the rider checked against the weights just calibrated. If this gives an appreciable correction for the rider, the individual weighings must be corrected for this error and all values recomputed. If this should change the rider correction by a significant amount, the weighings and computations must again be corrected, and the process repeated, if necessary, till no further change results.

XI. EFFECT OF WEIGHING METHODS

The method of weighing has no effect whatever on the scheme of calibration given herein. Any method that gives the desired accuracy may be used, and, therefore, it is often best to use the most familiar method, since this allows greater attention to be given to those details of manipulation that help to increase the reliability of the weighings. In general, it may not be found worth while to use the most precise weighing methods, such as methods designed to eliminate the effect of inequality of the arms of the balance beam, since these methods will be of no additional value except as they tend to increase the general accuracy in other ways.

In methods that use the deflection of the pointer in place of bringing the equilibrium position back to the zero point, it should be noted that additional small weights must be provided for determining the sensibility of the balance. The values of these weights must be known unless they are determined by the method of successive approximations as described for the determination of the value of the rider in the section on relative values.

Unless the weights under test and the "auxiliary weights" are very nearly equal, or the balance is not very sensitive, there must also be small, known weights or a known rider to make up the differences between the "auxiliary weights" and the weights being tested.

When the balance is intrinsically reliable enough a definite increase in accuracy can often be obtained by attention to such matters as not getting the hands far into the balance case, not breathing into the case, keeping sources of heat radiation, such as the face and hands, as far removed as possible and as symmetrically located as possible with reference to the beam.

A compromise must always be made between allowing a long time after closing the balance case for temporary air currents and temperature disturbances to subside, and working rapidly in order to avoid the drifting of zero, the change in ratio of arms and similar changes that always go on to a greater or less extent depending

chiefly on the constancy of temperature and humidity. Some weighing methods are faster than others, or require more or less handling of the balance. The choice of a weighing method depends, therefore, somewhat on familiarity and somewhat on the degree to which it affects the constancy of the weighing conditions. It is impossible to choose any one method as "the best," but a careful survey of individual circumstances may lead to one that is best for that particular occasion.

The choice of weighing method may, therefore, affect the accuracy of the results obtained, although it does not affect these systems of calibration.

XII. BUOYANT EFFECT OF THE AIR

The buoyant effect of the air is not an essential element in the scheme of calibration. These calibrations may be used for true mass values, or for values based on apparent weight in air against brass, or for values on any other basis. There must, however, be a proper uniformity between the basis on which the values of the standards are given and that on which the weighings are made or else the results will not have an intelligible meaning; and there must be a similar uniformity between the weighings of any one group and between each group and the value of N or M.

While a full discussion of this subject would be out of place here, a few simple facts will be given in the hope that they will help eliminate the need for any detailed consideration in a large number of cases.

Although it is true that in the calibration of most sets of weights, simplification or elimination of buoyancy corrections can be accomplished only at the expense of precision, yet rather high precision can be obtained with very simple allowance for this effect.

A degree of precision sufficient for a very large proportion of scientific and technical work is indicated in a general way by the precision to which the United States Bureau of Standards certifies the corrections for high-grade analytical weights. This certification is to the nearest unit in the decimal place indicated by the numeral 1, and to the nearest 5 or zero in the decimal place indicated by the numeral 5 in the statement below.

	mg
For weights of 100 g	0. 5
For weights of 50 g and 20 g	. 1
For weights of 10 g to 1 g, inclusive	. 05
For weights of 500 mg to 1 mg, inclusive	. 01

Most sets of high-grade laboratory weights are of brass down to 1 g, of platinum or materials of similar density from 500 mg to 50 mg or to 20 mg, and of aluminum below this. To the precision indicated above, such sets may be tested according to apparent weight against

brass standards in air without making allowance for the buoyant effect of the air, provided the values used for the standards are all computed on this basis. Apparent values determined in this way would be reliable to about the precision indicated, under all ordinary variations in the density of the atmosphere even up to an altitude of about 5,000 feet above sea level. It should be carefully noted, however, that weights must be both used and tested on the same basis in order to obtain this accuracy.

The calibration of sets in this manner should help rather than hinder the making of proper buoyancy corrections when the weights are used to weigh other objects, because such buoyancy corrections should be made as though all of the weights were of brass.

For brass weights, these corrections, according to apparent weight in air against brass, are, of course, identical with the values for the mass or for the "weight in vacuo." But uncertainties as to the density of individual weights, especially uncertainty as to the effect of the cavity and adjusting material under the knob of most laboratory weights, make it impossible to be sure of the mass values much, if any, beyond the precision indicated above.

For weights below 1 g the simplest procedure is generally to calibrate groups 3, 4, and 5 as though all weights of these groups were of the same material as the 1 g standard. A buoyancy reduction term can then be applied to the correction for each of the weights to reduce it to the value based on true mass, or on "apparent mass," or on whatever basis is desired. The values of the most used reduction terms are given below.

When the standard (S 1 g) is of platinum, the values based on apparent mass as compared with brass standards in air ("apparent weight against brass") will be obtained by adding to the computed corrections the terms given in the column headed 21.5 to 8.4. The true mass values of platinum weights will be those calculated in the calibration, and the true mass values for aluminum weights will be found by adding the terms in the column headed 21.5 to 2.7. When the standard (S 1 g) is of brass, the true mass values of platinum weights will be found by subtracting the terms found in the column headed 21.5 to 8.4 and those for aluminum by adding the terms in the column headed 8.4 to 2.7.

Buoyancy reduction terms for weights of the densities indicated

[Densities in g/cm [3]; air density [1] 1 2 mg/ml]

Nominal value of weight in milligrams	21.5 to 8.4	21.5 to 2.7	8.4 to 2.7
	mg	*mg*	*mg*
500	+0.044		
200	.017		
100	.009		
50	.004		
20	.002	+0.008	+0.006
10	.001	.004	.003
5	.000	.002	.002
2	.000	.001	.001
1	.000	.000	.000

[1]At an altitude of a few thousand feet above sea level the reduction in air density will reduce the two largest terms by a few units in the last decimal place given. The difference between the cm [3] and the ml is negligible for this work.

XIII. SIGN AND USE OF CORRECTIONS

When the correction for a weight has been determined, as in the calibration outlined here, this correction is the amount that should be added algebraically to the result of a weighing in which the weight had been used as though it were correct. In other words, an object may be weighed just as though the weights were exactly correct, and to this (uncorrected) result should be added the sum of the corrections for the weights used in the weighing.

This method of handling corrections saves a great deal of time and danger of making mistakes because it avoids handling the long numbers obtained by calculating the value of the weight as a single number. This may be seen from the following example:

Weights used	Computations		
	Using value as single number	Using corrections (value in two terms)	
		g	*mg*
(20 g)	19.999 8	20	−0.2
(2 g)	2.000 1	2	+ .1
(1 g)	.999 9	1	− .1
(500 mg)	.500 06	.5	+ .06
(50 mg)	.049 97	.05	− .03
(20 mg)	.019 99	.02	− .01
Rider (4.7 mg)	.004 71	.004 7	+ .01
	23.574 5	23.574 7	− .2 =23.574 5

An examination of the above definition of "corrections" shows that when the mass of the weight is greater than its nominal value—that is, when the weight is too heavy—the correction must be positive, and vice versa.

XIV. DISTINGUISHING DUPLICATE WEIGHTS

Duplicate weights may be easily distinguished by marking them with one dot and two dots, respectively. These dots can be best put on by using a very small-ended punch. The end should be well rounded

and smooth in order not to perforate the sheet-metal weights, and it is generally best to impress the dots on sheet-metal weights from below, resting the weight on lead or on close-grained hardwood in order that the dots may be clearly shown on the upper side. Neither plated weights nor properly lacquered weights will be injured by shallow dots made with such a tool.

It is preferable to have both duplicates marked, since the sight of some mark is always a more certain identification than failure to see any mark.

XV. PEDAGOGICAL CONSIDERATIONS

For use in schools, the simple computation of results shown here has one serious disadvantage. This is the tendency blindly to follow a formula without understanding its derivation and meaning. An ability accurately, even though blindly, to follow a simple mathematical expression is worth securing, but there is danger of introducing too much of this, even when an effort is made to teach the derivation and meaning when the formula is first taken up. These computations have, perhaps, one advantage over some of the ones that commonly occur, in that no one is likely to be so foolish as to expect them to be learned. In this regard they are, perhaps, good material on which to practice the blind following of a formula. It may be that in this or in some modified form these calibrations may prove useful in teaching, but this would be strictly incidental. The methods given here were chosen merely for convenience and simplicity in use.

XVI. BIBLIOGRAPHY

Other short methods of calibration may be found in the following:

Note on a method of standardizing weights, T. W. Richards, Jour. Am. Chem. Soc., **22**, p. 144; March, 1900.

The standardization of weights, A. J. Hopkins, J. B. Zinn, and Harriet Rogers, Jour. Am. Chem. Soc., **42**, p. 2528; December, 1920.

A few textbooks, especially those on laboratory physics, contain calibration methods generally based on the one given in Physical Measurements by Kohlrausch.

More elaborate methods may be found in the following:

B. S. Circular No. 3, Design and Test of Standards of Mass, 3d ed; 1918.

On the Least Square Adjustment of Weighings, John F. Hayford in Appendix No. 10 to Report for 1892 of United States Coast and Geodetic Survey.

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